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Harnack's Inequality on Homogeneous Spaces

Abstract. We consider a homogeneous space $X = (X, d, m)$ of dimension $\nu \geq 1$ and a local regular Dirichlet form in $L^2(X, m)$. We prove that if a Poincaré inequality holds on every pseudo-ball $B(x, R)$ of X, then an Harnack's inequality can be proved on the same ball with local characteristic constant c_0 and c_1

1. Introduction and Results

We consider a *connected, locally compact topological space* X. We suppose that a *distance* d is defined on X and we suppose that the balls

$$B(x, r) = \{y \in X : d(x, y) < r\}, \qquad r > 0,$$

form a basis of open neighborhoods of $x \in X$. Moreover, we suppose that a (positive) Radon measure m is given on X, with supp$m = X$. The triple (X, d, m) is assumed to satisfy the following property: There exist some constants $0 < R_0 \leq +\infty, \nu > 0$ and $c_0 > 0$, such that

$$0 < c_0 \left(\frac{r}{R}\right)^{\nu} m(B(x, R)) \leq m(B(x, r)) \tag{1.1}$$

for every $x \in X$ and every $0 < r \leq R < R_0$. Such a triple (X, d, m) will be called a homogeneous space of dimension ν. We point out, however, that a given exponent ν occurring in (1.1) should be considered, more precisely, as an upper bound of the "homogeneous dimension", hence we should better call (X, d, m) a homogeneous space of dimension less or equal than ν. We further suppose that we are also given a *strongly local, regular, Dirichlet form* a in the Hilbert space $L^2(X, m)$ - in the sense of M. Fukushima [2], - whose domain in $L^2(X, m)$ we shall denote by $\mathcal{D}[a]$. Furthermore, we shall restrict our study to Dirichlet forms of diffusion type, that is to forms a that have the following *strong local property:* $a(u, v) = 0$ for every $u, v \in \mathcal{D}[a]$ with v constant on supp u. We recall that the following integral representation of the form a holds

$$a(u, v) = \int_X \mu(u, v)(dx), \qquad u, v \in \mathcal{D}[a],$$

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where $\mu(u,v)$ is a uniquely defined signed Radon measure on X, such that $\mu(d,d) \leq m$, with $d \in \mathcal{D}_{loc}[a]$: this last condition is fundamental for the existence of cut off functions associated to the distance. Moreover, the restriction of the measure $\mu(u,v)$ to any open subset Ω of X depends only on the restrictions of the functions u, v to Ω. Therefore, the definition of the measure $\mu(u,v)$ can be unambiguously extended to all m-measurable functions u, v on X that coincide $m - a.e.$ on every compact subset of Ω with some functions of $\mathcal{D}[a]$. The space of all such functions will be denoted by $\mathcal{D}_{loc}[a,\Omega]$. Moreover we denote by $\mathcal{D}[a,\Omega]$ the closure of $\mathcal{D}[a] \cap C_0[\Omega]$ in $\mathcal{D}[a]$. The homogeneous metric d and the energy form a associated to the *energy measure* μ, both given on a relatively compact open subset X_0 of X with $\overline{\Omega} \subset X_0$, are then assumed to be mutually related by the following basic assumption:

There exists an exponent $s > 2$ and constants c_1, $c_1 > 0$ and $k \geq 1$, such that [1]:

i): for every $x \in X_0$ and every $0 < R < R_0$ the following Poincaré inequality holds:

$$\int_{B(x,R)} \left|u - \bar{u}_{B(x,R)}\right|^2 dm \leq c_1 R^2 \int_{B(x,kR)} \mu(u,u)(dx) \tag{1.2}$$

for all $u \in D[a, B(x,kR)]$, where

$$\bar{u}_{B(x,R)} = \frac{1}{m(B(x,R))} \int_{B(x,R)} udm.$$

ii): By i), for every $x \in X_0$ and every $0 < R < R_0$, the following Sobolev inequality of exponent s can be proved:

$$\left(\frac{1}{m(B(x,R))} \int_{B(x,R)} |u|^s \, dm\right)^{\frac{1}{s}} \leq c_1 R \left(\int_{B(x,R)} \mu(u,u)(dx)\right)^{\frac{1}{2}}, \tag{1.3}$$

where $u \in \mathcal{D}[a, B(x,kR)]$ and supp $u \subset B(x,R)$. Instead of working with the fixed constants of (1.1) and(1.2, 1.3), we make this simple generalization

$$c_0 \to c_0(x) \qquad \text{and} \qquad c_1 \to c_1(r), \tag{1.4}$$

where $c_0(x), c_0^{-1}(x) \in L^{\infty}_{loc}(X_0)$ and $c_1(r)$ is a decreasing function of r. Then we assume:

iii):

$$\int_{B(x,R)} \left|u - \bar{u}_{B(x,R)}\right|^2 dm \leq c_1^2(R) \, R^2 \int_{B(x,kR)} \mu(u,u)(dx). \tag{1.5}$$

By iii) the following Sobolev type inequality may be proved

iv):

(1.6)

$$\left(\frac{1}{m(B(x,R))} \int_{B(x,R)} |u|^s \, dm\right)^{\frac{1}{s}} \leq \tau^3 c_1(R) \, R \left(\frac{1}{m(B(x,R))} \int_{B(x,R)} \mu(u,u)(dx)\right)^{\frac{1}{2}},$$

where u is as in i) and ii) and where we have defined $\tau = \left(\sup_{B(x,2R)} \frac{1}{c_0(x)} \right)^{\frac{1}{2}}$.

Our purpose will be the Harnack's inequality recovery for Dirichlet forms when the substitution (1.4) is performed.

Theorem 1 (Harnack)**.** *Let* (1.1) , (1.5), (1.6) *and* (1.4) *hold, and let u be a non-negative solution of* $a(u,v) = 0$*. Let $\mathcal{O}$ be an open subset of X_0 and $u \in \mathcal{D}_{loc}[\mathcal{O}]$, $\forall v \in \mathcal{D}_0[\mathcal{O}]$ with $B(x,r) \subset \mathcal{O}$, then*

$$ess \sup_{B_{\frac{1}{2}}} u \leq \exp \gamma \mu' \mu^2 \; ess \inf_{B_{\frac{1}{2}}} u,$$

where $\gamma \equiv \gamma(\nu, k)$, k a positive constant, $\mu' = \tau c_1\left(\frac{r}{2}\right)$ and $\mu = \tau^3 c_1\left(\frac{r}{2}\right)$. A standard consequence of the previous Theorem is the following

Corollary 1. *Suppose that*

$$\int_r^R e^{-\gamma\mu(x,\rho)} \frac{d\rho}{\rho} \to \infty \qquad for \qquad r \to 0 \tag{1.7}$$

then the solution is continuous in the point under consideration. In particular, if $\mu(x,\rho) \approx o\left(\log\log\frac{1}{\rho}\right)$, then there exists $c > 0$ such that

$$\underset{B(x,r)}{osc}\; u \leq c \frac{\left(\log\frac{1}{R}\right)}{\left(\log\frac{1}{r}\right)} \underset{B(x,R)}{osc}\; u.$$

From the point of view of partial differential equations these results can be applied to two important classes of operators on $\mathbb{R}^n$:

a): *Doubly Weighted uniformly elliptic operators* in divergence form with measurable coefficients, whose coefficient matrix $A = (a_{ij})$ satisfies

$$w(x)|\xi|^2 \leq \langle A\xi, \xi\rangle \leq v(x)|\xi|^2 .$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual dot product; w and v are weight functions, respectively belonging to A_2 and D_∞ such that the following *Poincaré* inequality

$$\left(\frac{1}{|v(B)|}\int_B |f(x) - f_B|^q \, v dx\right)^{\frac{1}{2}} \leq cr \left(\frac{1}{|w(B)|}\int_B |\nabla f|^2 \, w dx\right)^{\frac{1}{2}}$$

holds.

: *Doubly Weighted Hörmander type operators*[3], whose form is $L = X_k^*\left(\alpha^{hk}(x) X_h\right)$ where $X_h, h = 1, \ldots, m$ are smooth vector fields in $\mathbb{R}^n$ that satisfy the Hörmander condition and $\alpha = \left(\alpha^{hk}\right)$ is any symmetric $m \times m$ matrix of measurable functions on $\mathbb{R}^n$,such that

$$w(x)\sum_i \langle X_i, \xi\rangle^2 \leq \sum_{i,j} \alpha_{ij}(x)\,\xi_i\xi_j \leq v(x)\sum_i \langle X_i, \xi\rangle^2 ,$$

where $X_i\xi(x) = \langle X_i, \nabla\xi\rangle$, $i = 1, \ldots, m$, $\nabla\xi$ is the usual gradient of ξ and $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $\mathbb{R}^n$. Then the following *Poincaré*

inequality for vector fields

$$\left(\frac{1}{|v(B)|}\int_B |f(x)-f_B|^q\, vdx\right)^{\frac{1}{2}} \le cr\left(\frac{1}{|w(B)|}\int_B \left(\sum_j |\langle X_j,\nabla f(x)\rangle|^2\right)^{\frac{1}{2}} wdx\right)^{\frac{1}{2}},$$

holds, with $w\in A_2$ and $v\in D_\infty$.

2. Harnack's Inequality

We prove the Harnack's Inequality of Theorem (1) for a non negative solution $u\ge\delta>0$ and with a constant C independent of δ. The result of Theorem (1) is obtained passing to the limit $\delta\to 0$.

Lemma 1. *Assume that* (1.1), (1.5), (1.6) *and* (1.4) *hold.*

Let u be a non-negative subsolution of $a(u,v)=0$, $u\in\mathcal{D}_{loc}[\mathcal{O}]$, $\forall v\in\mathcal{D}_0[\mathcal{O}]$. Define $\tau=\left(\sup\limits_{B(x,2R)}\frac{1}{c_0(x)}\right)^{\frac{1}{2}}$ *then*

$$\left(\underset{B_\alpha}{ess\,\sup} u\right)^p \le \left(c\tau^3\frac{c_1\left(\frac{r}{2}\right)}{(1-\alpha)}\right)^{2\frac{\sigma}{\sigma-1}}\left(\frac{1}{m(B)}\int_B u^p m(dx)\right),$$

where $\sigma=\frac{q}{2}$, $p\ge 2$ and

$$q=\begin{cases}\frac{2\nu}{\nu-2} & if\ \nu>2\\ any\ value & if\ \nu\le 2\end{cases}$$

Proof. We prove the result for a bounded non negative subsolutions. Let $\beta\ge 1$ and $0<M<\infty$, define $H_M(t)=t^\beta$ for $t\in[0,M]$ and $H_M(t)=M^\beta+\beta M^{\beta-1}(t-M)$ for $t>M$. For fixed M define

$$\phi_k(x)=\eta(x)^2\int_0^{u_k(x)} H_M'(t)^2\,dt$$

and

$$a(u,\phi):=\int_X \mu(u,\phi)(dx)=\int_X \mu\left(u,\eta(x)^2\int_0^u H_M'(t)^2\,dt\right)(dx)\le 0,$$

then

$$\int_X \mu\left(u,\eta(x)^2\int_0^u H_M'(t)^2\,dt\right)(dx)=\int_X \mu\left(u,\eta(x)^2\right)\int_0^u H_M'(t)^2\,dt\,(dx)$$

$$+\int_X \mu\left(u,\int_0^u H_M'(t)^2\,dt\right)\eta(x)^2\,(dx)= \tag{2.1}$$

$$\int_X 2\eta\mu(u,\eta)\int_0^u H_M'(t)^2\,dt\,(dx)+\int_X \mu(u,u)\,H_M'(u)^2\,\eta(x)^2\,(dx)\le 0,$$

and therefore

$$\int\limits_{X} \mu(u,u)\, H_M^{'}(u)^2\, \eta(x)^2\,(dx) \leq 2 \int\limits_{X} \left| \eta\mu(u,\eta) \int\limits_{0}^{u} H_M^{'}(t)^2\, dt \right| (dx)\,. \tag{2.2}$$

Taking account of the inequality

$$2\,|fg|\,|\mu(u,v)| \leq f^2\mu(u,u) + g^2\mu(v,v)$$

we get

$$2\,|\mu(u,\eta)| \left| \eta \int\limits_{0}^{u} H_M^{'}(t)^2\, dt \right| (dx) \leq \frac{1}{2}\mu(u,u)\,\eta^2 H_M^{'}(u)^2 + 2\mu(\eta,\eta) \left[\frac{1}{H_M^{'}(u)} \int\limits_{0}^{u} H_M^{'}(t)^2\, dt \right]^2$$

and putting into(2.1) we have

$$\frac{1}{2}\int\limits_{X} \mu(u,u)\, H_M^{'}(u)^2\, \eta(x)^2\,(dx) \leq 2 \int\limits_{X} \mu(\eta,\eta) \left[\frac{1}{H_M^{'}(u)} \int\limits_{0}^{u} H_M^{'}(t)^2\, dt \right]^2 (dx)$$

$$\leq 2 \int\limits_{X} \mu(\eta,\eta) \left[u H_M^{'}(u) \right]^2 (dx)\,. \tag{2.3}$$

Let us consider η as a cut-off function s.t. for $\frac{1}{2} \leq s < t < 1$ $B(x,sr) \subset B(x,tr) \subset \vartheta \subset\subset X, \eta \equiv 0$ on $X - B(x,tr)\,, \eta \equiv 1$ on $B(x,sr)\,, 0 \leq \eta < 1$ on X. Then

$$\int\limits_{B_s} \mu(H_M(u)\,, H_M(u))\, \eta(x)^2\,(dx) \leq \int\limits_{\vartheta} \mu(H_M(u)\,, H_M(u))\, \eta(x)^2\,(dx)$$

$$\leq \tfrac{40}{(t-s)^2 r^2} \int\limits_{B_t} \left[u H_M^{'}(u) \right]^2 m(dx)\,.$$

With the result of (1.2), applied to $H_M(u)$, one gets

$$\left(\tfrac{1}{m(B_s)} \int\limits_{B_s} \left| H_M(u) - \bar{H}_M(u)_{B_s} \right|^q m(dx) \right)^{\frac{1}{q}} \leq c\tau^2 c_1(sr)\, sr \left[\tfrac{1}{m(B_s)} \int\limits_{B_s} \mu(H_M(u)\,, H_M(u)) \right]^{\frac{1}{2}}$$

$$\leq c\tau^2 c_1(sr)\, sr \left(\tfrac{40}{(t-s)^2 r^2} \tfrac{1}{m(B_s)} \int\limits_{B_t} \left[u H_M^{'}(u) \right]^2 m(dx) \right)^{\frac{1}{2}}$$

$$\leq c\tau^2 \sqrt{40} c_1(sr)\, \tfrac{s}{(t-s)} \left(\tfrac{m(B_t)}{m(B_s)} \right)^{\frac{1}{2}} \left(\tfrac{1}{m(B_t)} \int\limits_{B_t} \left[u H_M^{'}(u) \right]^2 m(dx) \right)^{\frac{1}{2}}$$

$$\leq c\tau^3 \sqrt{40} c_1(sr)\, \tfrac{s}{(t-s)} \left(\tfrac{t}{s} \right)^{\frac{\nu}{2}} \left(\tfrac{1}{m(B_t)} \int\limits_{B_t} \left[u H_M^{'}(u) \right]^2 m(dx) \right)^{\frac{1}{2}}.$$

The average of $H_M(u)$ on B_s is defined by:

$$\underset{B_s}{av} H_M(u) = \frac{1}{m(B_s)} \int\limits_{Bs} H_M(u)\, m(dx) \leq \tau^2 \left(\frac{t}{s}\right)^{\nu} \frac{2}{m(B_t)} \int\limits_{B_s} H_M(u)\, m(dx)$$

$$\leq \tau \left(\frac{t}{s}\right)^{\frac{\nu}{2}} \left(\frac{1}{m(B_t)} \int\limits_{B_t} \left[u H_M'(u) \right]^2 m(dx) \right)^{\frac{1}{2}}$$

Recall that $m(B_t) \leq \tau^2 m(B_s) \left(\frac{t}{s}\right)^{\nu}$ and $H_M(u) \leq u H_M'(u)$. Therefore

$$\left(\frac{1}{m(B_s)} \int\limits_{B_s} |H_M(u)|^q\, dm \right)^{\frac{1}{q}} = \left(\frac{1}{m(B_s)} \int\limits_{B_s} \left| H_M(u) - \bar{H}_M(u)_{B_s} + \bar{H}_M(u)_{B_s} \right|^q dm \right)^{\frac{1}{q}}$$

$$\leq c \left[\left(\frac{1}{m(B_s)} \int\limits_{B_s} \left| H_M(u) - \bar{H}_M(u)_{B_s} \right|^q dm \right)^{\frac{1}{q}} + \left(\frac{1}{m(B_s)} \int\limits_{B_s} \left| \bar{H}_M(u)_{B_s} \right|^q dm \right)^{\frac{1}{q}} \right]$$

$$\leq c\tau^3 \left(\frac{t}{s}\right)^{\frac{\nu}{2}} \left(\frac{c_1(sr)s}{(t-s)} + 1 \right) \left(\frac{1}{m(B_t)} \int\limits_{B_t} \left[u H_M'(u) \right]^2 m(dx) \right)^{\frac{1}{2}};$$

where in the last expression we have included all irrelevant constants into c. From the fact that $\frac{s}{(t-s)} + 1 \leq 2\frac{s}{(t-s)}$, $\frac{s}{(t-s)} \geq 1$. Therefore due to $uH_M'(u) \leq u\beta u^{\beta-1} = \beta u^{\beta}, H_M(u) \geq u^{\beta}_{\chi_{\{\tilde{u} \leq M\}}}$, we obtain by letting $M \to \infty$ that

$$\left(\frac{1}{m(B_s)} \int\limits_{B_s} u^{\beta q} dm \right)^{\frac{1}{q}} \leq c\tau^3 \left(\frac{t}{s}\right)^{\frac{\nu}{2}} \left(\frac{\beta c_1(sr)s}{(t-s)} \right) \left(\frac{1}{m(B_t)} \int\limits_{B_t} u^{2\beta} m(dx) \right)^{\frac{1}{2}}.$$

Raise both sides to the power $\frac{1}{\beta}$ and putting $2\beta = \tilde{r}$ and $q = 2\sigma$ we have

$$\left(\frac{1}{m(B_s)} \int\limits_{B_s} u^{\sigma r} dm \right)^{\frac{1}{\sigma \tilde{r}}} \leq c \left[\tau^3 \left(\frac{t}{s}\right)^{\frac{\nu}{2}} \left(\frac{\beta c_1(sr)s}{(t-s)} \right) \right]^{\frac{2}{\tilde{r}}} \left(\frac{1}{m(B_t)} \int\limits_{B_t} u^{\tilde{r}} m(dx) \right)^{\frac{1}{\tilde{r}}}.$$

Now, starting from fixed α and $p, \frac{1}{2} \leq \alpha < 1, p \geq 2$ iterate this inequality for t and s successive entries in the sequence $s_j = \alpha + \frac{(1-\alpha)}{(j+1)}, j = 0, 1 \ldots,$ and $\tilde{r}$ and $\sigma\tilde{r}$ successive entries in $\left\{ \sigma^j p \right\}$, recalling that $\sigma > 1$. If we denote $a_j = \frac{s_j}{s_{j+1} - s_j}$ the conclusion of the lemma is a consequence of the estimate of

$$\log \prod_{j=0}^{\infty} \left[c\tau^3 c_1(s_j r) \left(\frac{s_{j+1}}{s_j} \right)^{\frac{\nu}{2}} \left(\sigma^j p a_j \right) \right]^{\frac{2}{\sigma^j p}} = \sum_{j=0}^{\infty} \frac{2}{\sigma^j p} \log \left[c\tau^3 c_1(s_j r) \left(\sigma^j p a_j \right) \right]$$

where we have used the fact that $\frac{s_{j+1}}{s_j}=\frac{\alpha+\frac{(1-\alpha)}{(j+2)}}{\alpha+\frac{(1-\alpha)}{(j+1)}}\simeq 1$. Therefore

$$\textstyle\sum_{j=0}^{\infty}\frac{2}{\sigma^j p}\log\left(c\tau^3 c_1\left(s_j r\right)p\right)+\sum_{j=0}^{\infty}\frac{2}{\sigma^j p}\log\left(\sigma^j a_j\right)$$

$$\textstyle\leq\frac{2}{p}\frac{\sigma}{\sigma-1}\log\left(c\tau^3 c_1\left(\frac{r}{2}\right)p\right)+\sum_{j=0}^{\infty}\frac{2}{\sigma^j p}\log\sigma^j+\sum_{j=0}^{\infty}\frac{2}{\sigma^j p}\left[2\log\left(j+2\right)+\log\frac{1}{1-\alpha}\right]$$

$$\textstyle\leq\frac{2}{p}\frac{\sigma}{\sigma-1}\log\left(c\tau^3 c_1\left(\frac{r}{2}\right)p\right)+\sum_{j=0}^{\infty}\frac{2}{\sigma^j p}\log\sigma^j+\frac{2}{p}\frac{\sigma}{\sigma-1}\log\frac{1}{1-\alpha}+4\sum_{j=0}^{\infty}\frac{\log(j+2)}{\sigma^j p}$$

$$\textstyle\leq\frac{2}{p}\frac{\sigma}{\sigma-1}\log\left(\frac{cp}{1-\alpha}c_1\left(\frac{r}{2}\right)\tau^3\right)$$

and at the end

$$\prod_{j=0}^{\infty}\left[c\tau^3 c_1\left(s_j r\right)\left(\frac{s_{j+1}}{s_j}\right)^{\frac{\nu}{2}}\left(\sigma^j p a_j\right)\right]^{\frac{2}{\sigma^j p}}\leq\left(cc_1\left(\frac{r}{2}\right)\frac{\tau^3}{1-\alpha}\right)^{\frac{2}{p}\frac{\sigma}{\sigma-1}},$$

where we used the following inequalities $\log\left(a_j\right)\leq 2\log\left(j+2\right)+\log\frac{1}{1-\alpha}$, $p^{\frac{1}{p}}\leq c$ for $p\geq 2$. The general case can be obtained if we apply the above result for $p=2$ to the truncated subsolutions and we obtain that u is locally bounded. Coming back to the previous result, we obtain the Lemma (1) by an approximation by truncation.

Lemma 2. *With the same notation and hypothesis as in Lemma* (1) *and* $-\infty<p<+\infty$,

$$\left(\underset{B_\alpha}{ess\,\sup}u^p\right)\leq\left(\frac{c\tau^3 c_1\left(\frac{r}{2}\right)p+1}{(1-\alpha)}\right)^{\frac{2\sigma}{(\sigma-1)}}\left(\frac{1}{m\left(B_t\right)}\int_{B_t}u^p m\left(dx\right)\right).$$

Proof. It is sufficient to consider $-\infty<p<2$. Define $\phi\left(x\right)=\eta^2\left(x\right)u^\beta\left(x\right)$, $\eta\left(x\right)\geq 0$, $-\infty<\beta<+\infty$. Then

$$\textstyle 0\geq\int_X\mu\left(u,\phi\right)\left(dx\right)=\int_X\mu\left(u,\eta^2u^\beta\right)\left(dx\right)=\int_X\eta^2\mu\left(u,u^\beta\right)\left(dx\right)+\int_X u^\beta\mu\left(u,\eta^2\right)\left(dx\right)$$

$$\textstyle =\int_X\beta u^{\beta-1}\eta^2\mu\left(u,u\right)\left(dx\right)+\int_X u^\beta 2\eta\mu\left(u,\eta\right)\left(dx\right).$$

Observe that for $\beta\neq-1$, we can write

$$\frac{4\beta}{\left(\beta+1\right)^2}\int_X\eta^2\mu\left(u^{\frac{\beta+1}{2}},u^{\frac{\beta+1}{2}}\right)\left(dx\right)=\int_X\beta u^{\beta-1}\eta^2\mu\left(u,u\right)\left(dx\right)$$

and

$$2\int_X u^{\frac{\beta-1}{2}}u^{\frac{\beta+1}{2}}\eta\mu\left(\eta,u\right)\left(dx\right)=\frac{4}{\beta+1}\int_X\mu\left(u^{\frac{\beta+1}{2}},\eta\right)u^{\frac{\beta+1}{2}}\eta\left(dx\right).$$

Then,

$$-\int_X\beta u^{\beta-1}\eta^2\mu\left(u,u\right)\left(dx\right)\leq 2\int_X u^\beta\eta\mu\left(\eta,u\right)\left(dx\right),$$

but

$$\textstyle\int_X\beta u^{\beta-1}\eta^2\mu\left(u,u\right)\left(dx\right)=\frac{4\beta}{(\beta+1)^2}\int_X\eta^2\mu\left(u^{\frac{\beta+1}{2}},u^{\frac{\beta+1}{2}}\right)\left(dx\right)$$

$$\textstyle\leq\frac{4}{(\beta+1)}\int_X\mu\left(u^{\frac{\beta+1}{2}},\eta\right)u^{\frac{\beta+1}{2}}\eta\left(dx\right).$$

Taking absolute values gives

$$\frac{\beta}{(\beta+1)}\int_X \eta^2\left|\mu\left(u^{\frac{\beta+1}{2}},u^{\frac{\beta+1}{2}}\right)\right|(dx)\le\int_X u^{\frac{\beta+1}{2}}\eta\mu\left(u^{\frac{\beta+1}{2}},\eta\right)(dx)\,; \tag{2.4}$$

recalling the fundamental inequality $2\left|fg\right|\left|\mu\left(u,v\right)\right|\;\le\;f^2\mu\left(u,u\right)+g^2\mu\left(v,v\right),$ we have

$$2\left|fg\right|\left|\mu\left(u,v\right)\right|=\left|u^{\frac{\beta+1}{2}}\eta\right|\left|\mu\left(u^{\frac{\beta+1}{2}},\eta\right)\right|\le\frac{\left|\beta\right|}{2\left|\beta+1\right|}\mu\left(u^{\frac{\beta+1}{2}},u^{\frac{\beta+1}{2}}\right)\eta^2+\frac{\left|\beta+1\right|}{2\left|\beta\right|}u^{\beta+1}\mu\left(\eta,\eta\right).$$

Then, from (2.4) it follows that

$$\frac{\left|\beta\right|}{2\left|\beta+1\right|}\int_X\eta^2\left|\mu\left(u^{\frac{\beta+1}{2}},u^{\frac{\beta+1}{2}}\right)\right|(dx)\le\frac{\left|\beta+1\right|}{2\left|\beta\right|}\int_X u^{\beta+1}\mu\left(\eta,\eta\right)(dx)\,,$$

that is

$$\int_X\eta^2\left|\mu\left(u^{\frac{\beta+1}{2}},u^{\frac{\beta+1}{2}}\right)\right|(dx)\le\left(\frac{\left|\beta+1\right|}{2\left|\beta\right|}\right)^2\int_X u^{\beta+1}\mu\left(\eta,\eta\right)(dx)\,.$$

This is the same as (1.2); beginning from the Sobolev inequality applied to $u^{\frac{\beta+1}{2}}$ with the same meaning and definition of the cut-off functions η, one gets

$$\left(\tfrac{1}{m(B_s)}\int\limits_{B_s}\left|u^{\frac{\beta+1}{2}}-\bar{u}^{\frac{\beta+1}{2}}{}_{B_s}\right|^q m\left(dx\right)\right)^{\frac{1}{q}}\le c\tau^2c_1\left(sr\right)sr\left[\tfrac{1}{m(B_s)}\int\limits_{B_s}\mu\left(u^{\frac{\beta+1}{2}},u^{\frac{\beta+1}{2}}\right)(dx)\right]^{\frac{1}{2}}$$

$$\le c\tau^2c_1\left(sr\right)sr\left(\tfrac{|\beta+1|}{2|\beta|}\right)\left(\int_{B_s}u^{\beta+1}\mu\left(\eta,\eta\right)(dx)\right)^{\frac{1}{2}}$$

$$\le c\tau^2\,\tfrac{c_1(sr)s}{t-s}\left(\tfrac{|\beta+1|}{2|\beta|}\right)\left(\tfrac{1}{m(B_s)}\int_{B_s}u^{\beta+1}m\left(dx\right)\right)^{\frac{1}{2}}$$

$$\le c\tau^2\frac{c_1\left(sr\right)s}{t-s}\left(\frac{\left|\beta+1\right|}{2\left|\beta\right|}\right)\left[\tau^2\left(\frac{t}{s}\right)^{\nu}\right]^{\frac{1}{2}}\left(\frac{1}{m\left(B_t\right)}\int_{B_t}u^{\beta+1}m\left(dx\right)\right)^{\frac{1}{2}}. \tag{2.5}$$

Evaluating the average of $u^{\frac{\beta+1}{2}}$ on B_s, one gets

$$\underset{B_s}{avu}^{\frac{\beta+1}{2}}=\frac{1}{m\left(B_s\right)}\int_{B_s}u^{\frac{\beta+1}{2}}m\left(dx\right)\le\frac{\tau^2}{m\left(B_t\right)}\left(\frac{t}{s}\right)^{\nu}\int_{B_t}u^{\frac{\beta+1}{2}}m\left(dx\right). \tag{2.6}$$

By Hölder inequality

$$\frac{\tau^2}{m\left(B_t\right)}\left(\frac{t}{s}\right)^{\nu}\int_{B_t}u^{\frac{\beta+1}{2}}m\left(dx\right)\le\left[\tau^2\left(\frac{t}{s}\right)^{\nu}\right]^{\frac{1}{2}}\left[\frac{1}{m\left(B_t\right)}\int_{B_t}u^{\beta+1}m\left(dx\right)\right]^{\frac{1}{2}}$$

Putting together (2.5) and (2.6), we see that

$$\left(\tfrac{1}{m(B_s)}\int\limits_{B_s}\left|u^{\frac{\beta+1}{2}}\right|^q m\left(dx\right)\right)^{\frac{1}{q}}=\left(\tfrac{1}{m(B_s)}\int\limits_{B_s}\left|u^{\frac{\beta+1}{2}}-\bar{u}^{\frac{\beta+1}{2}}{}_{B_s}+\bar{u}^{\frac{\beta+1}{2}}{}_{B_s}\right|^q m\left(dx\right)\right)^{\frac{1}{q}}$$

$$\le c\left[\left(\tfrac{1}{m(B_s)}\int\limits_{B_s}\left|u^{\frac{\beta+1}{2}}-\bar{u}^{\frac{\beta+1}{2}}{}_{B_s}\right|^q m\left(dx\right)\right)^{\frac{1}{q}}+\left(\tfrac{1}{m(B_s)}\int\limits_{B_s}\left|\bar{u}^{\frac{\beta+1}{2}}{}_{B_s}\right|^q m\left(dx\right)\right)^{\frac{1}{q}}\right]$$

$$\le\left(c\tau^3\frac{c_1\left(sr\right)s}{t-s}\frac{\left|\beta+1\right|}{2\left|\beta\right|}+1\right)\left(\frac{t}{s}\right)^{\frac{\nu}{2}}\left(\frac{1}{m\left(B_t\right)}\int_{B_t}u^{\beta+1}m\left(dx\right)\right)^{\frac{1}{2}}.$$

Setting $\beta+1=\tilde{r}$ and $q=2\sigma$, we see that for any r with $-\infty<\tilde{r}\leq 2, \tilde{r}\neq 0,1$

$$\left(\tfrac{1}{m(B_s)}\int\limits_{B_s} u^{|\tilde{r}|\sigma} m\left(dx\right)\right)^{\frac{1}{|\tilde{r}|\sigma}}$$

$$\leq \left\{\left(c\tau^3 \frac{c_1\left(sr\right)s}{t-s}\frac{|\tilde{r}|}{2\left|\tilde{r}-1\right|}+1\right)\left(\frac{t}{s}\right)^{\frac{\nu}{2}}\right\}^{\frac{2}{|\tilde{r}|}}\left(\frac{1}{m\left(B_t\right)}\int_{B_t} u^{\beta+1} m\left(dx\right)\right)^{\frac{1}{|\tilde{r}|}}.$$

We use the iteration argument with any fixed p as a starting value of $\tilde{r}$, with $-\infty<p\leq 2, p\neq 0,1$.

a): $p<0$ $\tilde{r}=\sigma^j p\rightarrow -\infty$

$$\prod_{j=0}^{\infty}\left\{\left(c\tau^3 c_1\left(s_j r\right)a_j\frac{\sigma^j\left|p\right|}{2\left|\sigma^j p-1\right|}+1\right)\left(\frac{s_{j+1}}{s_j}\right)^{\frac{\nu}{2}}\right\}^{\frac{2}{\sigma^j|p|}}$$

$$\leq \prod_{j=0}^{\infty}\left\{\left(c\tau^3 c_1\left(s_j r\right)a_j\sigma^j\left|p\right|+1\right)\right\}^{\frac{2}{\sigma^j|p|}}$$

Then, after the conversion to the log of the previous quantity we have

$$\log\textstyle\prod_{j=0}^{\infty}\left\{\left(c\tau^3 c_1\left(s_j r\right)a_j\sigma^j\left|p\right|+1\right)\right\}^{\frac{2}{\sigma^j|p|}}=\sum_{j=0}^{\infty}\frac{2}{\sigma^j|p|}\log\left\{\left(c\tau^3 c_1\left(s_j r\right)a_j\sigma^j\left|p\right|+1\right)\right\}$$

$$=\textstyle\sum_{j=0}^{\infty}\frac{2}{\sigma^j|p|}\left[\log\left(c\frac{a_j\sigma^j}{c_1(s_j r)\tau^3|p|+1}\right)+\log\left(\tau^3 c_1\left(s_j r\right)\left|p\right|+1\right)\right]$$

$$\leq \textstyle\frac{2\sigma}{(\sigma-1)|p|}\log\left(\tau^3 c_1\left(\frac{r}{2}\right)\left|p\right|+1\right)+\sum_{j=0}^{\infty}\frac{4}{\sigma^j|p|}\left[\log\left(c\frac{(j+2)\sigma^j}{c_1(s_j r)\tau^3|p|+1}\right)\right]$$

$$+\sum_{j=0}^{\infty}\frac{2}{\sigma^j\left|p\right|}\left[\log\left(c\frac{\left(j+2\right)\sigma^j}{\left(c_1\left(s_j r\right)\tau^3\left|p\right|+1\right)\left(1-\alpha\right)}\right)\right]\leq\frac{2\sigma}{\left(\sigma-1\right)\left|p\right|}\log\left(\frac{c\tau^3 c_1\left(\frac{r}{2}\right)\left|p\right|+1}{\left(1-\alpha\right)}\right)$$

b): $0<p<2$

$$\prod_{j=0}^{\infty}\left\{\left(c\tau^3 c_1\left(\frac{r}{2}\right)a_j\sigma^j p+1\right)\right\}^{\frac{2}{\sigma^j p}}\leq\left(\frac{c\tau^3 c_1\left(\frac{r}{2}\right)p+1}{\left(1-\alpha\right)}\right)^{\frac{2\sigma}{(\sigma-1)p}}$$

Lemma 3. *Let the hypothesis of Lemma (2) hold, except that now u is a non-negative supersolution of $a\left(u,v\right)=0$. For $\frac{1}{2}\leq\alpha<1$, define $k=k\left(\alpha,u\right)$ by*

$$\log k=\frac{1}{m\left(B_\alpha\right)}\int\limits_{B_\alpha}\left(\log u\right)dm.$$

Then for $\lambda>0$,

$$m\left(\left\{x\in B_\alpha:\left|\log\frac{u\left(x\right)}{k}\right|>\lambda\right\}\right)\leq\frac{c\tau c_1\left(r\right)}{\left(1-\alpha\right)}\frac{1}{\lambda}m\left(B_\alpha\right).$$

Proof. Letting $\phi = \frac{\eta^2}{u}$, it follows that $\|\phi\|_0$ is bounded. We observe that

$$0 \leq \int_X \mu(u,\phi)(dx) = \int_X \mu\left(u, \frac{\eta^2}{u}\right)(dx) =$$

$$\int_X \eta^2 \mu\left(u, \tfrac{1}{u}\right)(dx) + \int_X \tfrac{1}{u}\mu\left(u,\eta^2\right)(dx) = \int_X \eta^2 \mu(u,u)\left(-\tfrac{1}{u^2}\right)(dx) + \int_X \tfrac{1}{u}\mu(u,\eta)\, 2\eta\,(dx)$$

This implies that

$$\int_X \eta^2 \mu(u,u)\left(\frac{1}{u^2}\right)(dx) \leq 2\int_X \frac{1}{u}\mu(u,\eta)\,\eta\,(dx)$$

Taking the absolute values and recognizing the gradient of the log to the left side, we have

$$\int_X \eta^2 \mu(\log u, \log u)(dx) \leq 2\int_X \left|\frac{1}{u}\mu(u,\eta)\,\eta\right|(dx).$$

Recalling the usual fundamental inequality, we obtain that

$$2\left|\frac{1}{u}\eta\right| |\mu(u,\eta)| \leq \frac{1}{2}\left(\frac{\eta}{u}\right)^2 \mu(u,u) + 2\mu(\eta,\eta),$$

then $\frac{1}{2}\int_X \eta^2\mu(\log u,\log u)(dx) \leq 2\int_X \mu(\eta,\eta)(dx)$. If we choose as a cut-off function $\eta = 1$ on B_α s.t. $|\mu(\eta,\eta)| \leq \frac{b}{(1-\alpha)^2 r^2}$, where b is a constant, we get

$$\int\limits_{B_\alpha} |\mu(\log u,\log u)|(dx) \leq \frac{b}{(1-\alpha)^2 r^2}\int\limits_{B_\alpha} m(dx) = \frac{bm(B_\alpha)}{(1-\alpha)^2 r^2}.$$

By the Poincaré inequality,

$$\int\limits_{B_\alpha} \left|\log u - \overline{\log u}\right|^2 m(dx) \leq cc_1^2(r)\, r^2 \int\limits_{B_{k\alpha}} \mu(\log u,\log u)(dx)$$

$$\leq bcc_1^2(r)\frac{m(B_{k\alpha})}{(1-\alpha)^2}.$$

It follows that

$$av_{B_\alpha}(\log u) = \frac{1}{m(B_\alpha)}\int\limits_{B_\alpha} (\log u)\, m(dx) \to \log k.$$

Therefore, by including in c every other inessential constant

$$\int\limits_{B_\alpha} |\log u - \log k|^2 m(dx) \leq cc_1^2(r)\frac{m(B_{k\alpha})}{(1-\alpha)^2}.$$

By Chebyshev's inequality, for $\lambda > 0$,

$$m\left(\left\{x \in B_\alpha : \left|\log\tfrac{u(x)}{k}\right| > \lambda\right\}\right) \leq \tfrac{1}{\lambda}\int\limits_{B_\alpha} \left|\log\tfrac{u}{k}\right| m(dx)$$

$$\leq \tfrac{1}{\lambda}\left(\int\limits_{B_\alpha} \left|\log\tfrac{u}{k}\right|^2 m(dx)\right)^{\frac{1}{2}} m(B_\alpha)^{\frac{1}{2}} \leq \tfrac{1}{\lambda}\left(cc_1^2(r)\tfrac{m(B_{k\alpha})}{(1-\alpha)^2}\right)^{\frac{1}{2}} m(B_\alpha)^{\frac{1}{2}}$$

$$\leq \frac{1}{\lambda}\tau\left(\frac{k^\nu cc_1^2(r)}{(1-\alpha)^2}m(B_\alpha)\right)^{\frac{1}{2}} m(B_\alpha)^{\frac{1}{2}} \leq \frac{1}{\lambda}\tau\left(\frac{cc_1^2(r)}{(1-\alpha)^2}\right)^{\frac{1}{2}} m(B_\alpha).$$

Lemma 4 (Moser)**.** [5] *Let $c_1(r), c_0(x)$ and $f(x)$ be non-negative bounded functions on a ball B and in particular let $c_0(x)$ belong to $L^\infty_{loc}(X_0)$ together with $c_0^{-1}(x)$; $c_1(r)$ be a decreasing function of r . Assume that there are constants D, d so that*

(a): $\operatorname{esssup}_{B_s}(f^p) \le \left(\frac{\mu}{t-s}\right)^d \frac{1}{m(B_t)} \int_{B_t} f^p m(dx)$ *for all* s, t, p *with* $0 < p < \frac{1}{\mu}$ *and* $\frac{1}{2} \le s < t \le 1$, *where* $\mu = \tau^3 c_1\left(\frac{r}{2}\right)$

(b): $m\left(\left\{x \in B_{\frac{1}{2}} : \log f(x) > \lambda\right\}\right) \le c\frac{\mu'}{\lambda} m(B)$, $\forall \lambda > 0$, *where* $\mu = \tau c_1\left(\frac{r}{2}\right)$.

Then, there exists a constant $\gamma \equiv \gamma(c,d)$ s.t.,

$$\operatorname{ess\,sup}_{B\frac{1}{2}} f \le \exp \gamma \mu^2 \mu'. \tag{2.7}$$

Proof. Replacing f with f^μ and λ with $\lambda\mu$, we simplify the hypothesis to the case $\mu = 1$. Similarly, we may assume that $m(B) = 1$ and the result will be valid for $\mu = 1$ too. Define $\varphi(s) = \sup_{B_s} \log f$ for $\frac{1}{2} \le s < 1$, which is a nondecreasing function. Decompose B_t into the sets where $\log f > \frac{1}{2}\varphi(t)$ and where $\log f \le \frac{1}{2}\varphi(t)$ and accordingly estimate the integral

$$\int_{B_t} f^p m(dx) \le e^{p\varphi} \frac{2c}{\varphi} + e^{p\frac{\varphi}{2}} \tag{2.8}$$

where $\varphi = \varphi(r)$ and we have used

$$m\left(\left\{x \in B_{\frac{1}{2}} : \log f(x) > \lambda\right\}\right) \le \frac{c}{\lambda} \tag{2.9}$$

and the normalization $m(B) = 1$. We choose p so that the two terms on the r.h.s. are equal, i.e.

$$p = \frac{2}{\varphi} \log\left(\frac{\varphi}{2c}\right), \tag{2.10}$$

provided that this quantity is less than 1 so that $0 < p < \mu^{-1} = 1$ holds. $\frac{2}{\varphi}\log\left(\frac{\varphi}{2c}\right) < 1$ means that $c > \frac{\varphi}{2}e^{-\frac{\varphi}{2}}$, but $\frac{\varphi}{2}e^{-\frac{\varphi}{2}}$ assumes its maximum at the value $\varphi = 2$ and this means that $\max\left(\frac{\varphi}{2}e^{-\frac{\varphi}{2}}\right) = e^{-1}$. Therefore, if $c > e^{-1}$ then $p < 1$; otherwise this requires that $\varphi = \varphi(r) > c_1$ depending only by c. In that case, we have

$$\int_{B_t} f^p m(dx) \le 2e^{p\frac{\varphi}{2}} \tag{2.11}$$

and hence, by Hp a)

$$\varphi(s) < \frac{1}{p} \log\left(2\left(\frac{\mu}{t-s}\right)^d e^{p\frac{\varphi}{2}}\right) = \frac{1}{p}\log\left(2\left(\frac{\mu}{t-s}\right)^d\right) + \frac{\varphi(t)}{2} \tag{2.12}$$

and by (2.10)

$$\varphi(s) < \frac{\varphi(t)}{2}\left\{\frac{\log\left(2\left(\frac{\mu}{t-s}\right)^d\right)}{\log\left(\frac{\varphi}{2c}\right)} + 1\right\}. \tag{2.13}$$

If $\varphi(t)$ is so large that the first term in the parentheses is less than $\frac{1}{2}$, i.e. if

(2.14)

$$\log\left(2\left(\frac{\mu}{t-s}\right)^{d}\right) < \frac{1}{2}\log\left(\frac{\varphi}{2c}\right) \Longrightarrow \left(2\left(\frac{\mu}{t-s}\right)^{d}\right)^{2} < \frac{\varphi}{2c} \Longrightarrow \varphi(t) > 8c\left(\frac{\mu}{t-s}\right)^{2d}$$

then clearly $\varphi(s) < \frac{3}{4}\varphi(t)$. Anyway, let us distinguish the case when $p > 1$ or (2.14) fail. This means that:

1. if $p > 1$, but (2.14) is still valid, then $c < \frac{\varphi}{2}e^{-\frac{\varphi}{2}}$; therefore

$$c < 4c\left(\frac{\mu}{t-s}\right)^{2d} e^{-\frac{\varphi}{2}} \Longrightarrow ce^{\frac{\varphi}{2}} < 4c\left(\frac{\mu}{t-s}\right)^{2d}, \text{ but } ce^{\frac{\varphi}{2}} > \varphi \Longrightarrow \varphi < 4c\left(\frac{\mu}{t-s}\right)^{2d}$$

2. if (2.14) is violated $\Longrightarrow \varphi(t) < 8c\left(\frac{\mu}{t-s}\right)^{2d}$.

In any case $\varphi(t) < 8c\left(\frac{\mu}{t-s}\right)^{2d}$. Since $\varphi(s) \leq \varphi(t)$, we have in both cases that

$$\varphi(s) \leq \frac{3}{4}\varphi(t) + \frac{\gamma_1}{(t-s)^{2d}}, \tag{2.15}$$

where $\gamma_1 \equiv \gamma_1(c, d)$. For every sequence $\frac{1}{2} \leq s_0 \leq s_1 \leq \ldots\ldots \leq s_k \leq 1$, we iterate the inequality (2.15) :

Step 1)

$$\varphi(s_0) < \frac{3}{4}\varphi(s_1) + \frac{\gamma_1\mu^2}{(s_1-s_0)^{2d}}$$

1.

$$\varphi(s_1) < \frac{3}{4}\varphi(s_2) + \frac{\gamma_1\mu^2}{(s_2-s_1)^{2d}}, \text{ but } \frac{4}{3}\varphi(s_0) - \frac{4}{3}\frac{\gamma_1\mu^2}{(s_1-s_0)^{2d}} < \varphi(s_1) \text{ then}$$

$$\tfrac{4}{3}\varphi(s_0) - \tfrac{4}{3}\tfrac{\gamma_1\mu^2}{(s_1-s_0)^{2d}} < \tfrac{3}{4}\varphi(s_2) + \tfrac{\gamma_1\mu^2}{(s_2-s_1)^{2d}}$$

$$\Longrightarrow \varphi(s_0) < \left(\tfrac{3}{4}\right)^2\varphi(s_2) + \gamma_1\mu^2\left(\tfrac{1}{(s_1-s_0)^{2d}} + \tfrac{3}{4}\tfrac{1}{(s_2-s_1)^{2d}}\right)$$

1.

$$\varphi(s_2) < \tfrac{3}{4}\varphi(s_3) + \tfrac{\gamma_1\mu^2}{(s_3-s_2)^{2d}}$$

$$\Longrightarrow \left(\tfrac{4}{3}\right)^2\varphi(s_0) - \gamma_1\mu^2\left(\tfrac{4}{3}\right)^2\left(\tfrac{1}{(s_1-s_0)^{2d}} + \tfrac{3}{4}\tfrac{1}{(s_2-s_1)^{2d}}\right) < \varphi(s_2)$$

$$\Longrightarrow \varphi(s_0) < \left(\tfrac{3}{4}\right)^3\varphi(s_3) + \gamma_1\mu^2\left(\tfrac{1}{(s_1-s_0)^{2d}} + \tfrac{3}{4}\tfrac{1}{(s_2-s_1)^{2d}} + \left(\tfrac{3}{4}\right)^2\tfrac{1}{(s_2-s_1)^{2d}}\right)$$

$$\varphi(s_0) < \left(\frac{3}{4}\right)^k\varphi(s_k) + \gamma_1\mu^2\sum_{j=0}^{k-1}\left(\frac{3}{4}\right)^j\frac{1}{(s_{j+1}-s_j)^{2d}}.$$

By monotonicity, we have $\varphi(s_k) \leq \varphi(s_1) < \infty$ and letting $k \to \infty$, we obtain

$$\varphi\left(\frac{1}{2}\right) \leq \gamma_1\mu^2\sum_{j=0}^{\infty}\left(\frac{3}{4}\right)^j\frac{1}{(s_{j+1}-s_j)^{2d}}.$$

The r.h.s. will converge if we choose, for example, $s_j = 1 - \frac{1}{2(1+j)}$

$$\Longrightarrow \varphi\left(\tfrac{1}{2}\right) \leq \gamma_1 \mu^2 \sum_{j=0}^{\infty} \left(\tfrac{3}{4}\right)^j \left(\tfrac{1}{2(j+2)(j+1)}\right)^{2d} = \gamma\mu^2$$

$$\Longrightarrow \quad \sup_{B_{\frac{1}{2}}} f \leq e^{\gamma\mu^2}.$$

Proof of Harnack Inequality.(for $\delta > 0$)Let (1.1) , (1.2) and (1.4) hold, u be a non-negative solution of $a(u, v) = 0$, $u \in \mathcal{D}_{loc}[\mathcal{O}]$, $\forall v \in \mathcal{D}_0[\mathcal{O}]$. We wish to apply Lemma (4) to both u/k and k/u, with k defined by

$$\log k = \frac{1}{m\left(B_{\frac{3}{2}\alpha}\right)} \int\limits_{\frac{3}{2}\alpha} (\log u)\, m(dx)\,. \tag{2.16}$$

Assumption (b) of the lemma holds by applying Lemma (3)to $B_{2\alpha} \subset \mathcal{O}$; because of the presence of an absolute value in the same Lemma and since $\log(u/k) = -\log(k/u)$, assumption (b) holds for both u/k and k/u. Assumption (a) holds by Lemma (1) applied to u/k with $d = \frac{\sigma}{\sigma-1}$. We obtain from (4) both

$$\underset{B_{\frac{1}{2}}}{ess} \sup(u/k) \leq e^{\gamma\mu^2\mu'}, \qquad \underset{B_{\frac{1}{2}}}{ess} \sup(k/u) \leq e^{\gamma\mu^2\mu'}$$

and the result follows by taking the product of these estimates, that is

$$\underset{B_{\frac{1}{2}}}{ess} \sup u \leq e^{\gamma\mu^2\mu'} \underset{B_{\frac{1}{2}}}{ess} \inf u.$$

Proof of Corollary. We may assume without loss of generality that $R \leq R_0/4$, with $R_0 < 1$ and $r \leq \frac{R}{4}$ in such a way that $B(x, 4R) \subset B(x, R_0) \subset \mathcal{O}$. Let us define

$$M_R = \sup_{B_R} u \qquad m_R = \inf_{B_R} u \qquad M_r = \sup_{B_r} u \qquad m_r = \inf_{B_r} u.$$

Then by applying Harnack inequality to the functions $M_R - u$, $u - m_R$ in B_r, we obtain

$$M_R - u \leq \sup_{B_r}(M_R - u) \leq e^{\gamma\mu(x,r)} \inf_{B_r}(M_R - u) = M_R - M_r$$

and

$$u - m_R \leq \sup_{B_r}(u - m_R) \leq e^{\gamma\mu(x,r)} \inf_{B_r}(u - m_R) = m_r - m_R$$

Hence by addition,

$$M_R - m_R \leq e^{\gamma\mu(x,r)}(M_R - M_r + m_r - m_R)$$

so that, writing

$$\omega(r) = \underset{B_r}{osc}\, u = M_r - m_r$$

we have

$$\omega(r) \leq \left(1 - e^{-\gamma\mu(x,r)}\right)\omega(R)\,, \tag{2.17}$$

with $\omega(R) = \omega(4r)$. The application of Lemma 6.5 of [6], gives the following inequality

$$\omega(r) \leq \exp\left(-c\int_{\frac{r}{4}}^{R} e^{-\gamma\mu(x,\rho)}\frac{d\rho}{\rho}\right)\omega(R). \tag{2.18}$$

Let suppose that $\mu(x,\rho) \approx o\left(\log\log\frac{1}{\rho}\right)$, then

$$\omega(r) \leq \exp\left(-c\int_{\frac{r}{4}}^{r} e^{-\gamma\mu(x,\rho)}\frac{d\rho}{\rho} - c\int_{r}^{R} e^{-\gamma\mu(x,\rho)}\frac{d\rho}{\rho}\right)\omega(R),$$

$$\Longrightarrow \omega(r) \leq \exp\left(-\int_{\frac{r}{4}}^{r}\frac{1}{\log\frac{1}{\rho}}\frac{d\rho}{\rho} - \int_{r}^{R}\frac{1}{\log\frac{1}{\rho}}\frac{d\rho}{\rho}\right)\omega(R),$$

that is,

$$\omega(r) \leq c\frac{\left(\log\frac{1}{R}\right)}{\left(\log\frac{1}{r}\right)}\omega(R). \tag{2.19}$$

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